# The size of trees: Exploring biological scaling

Benjamin Blonder

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## 1 Introduction

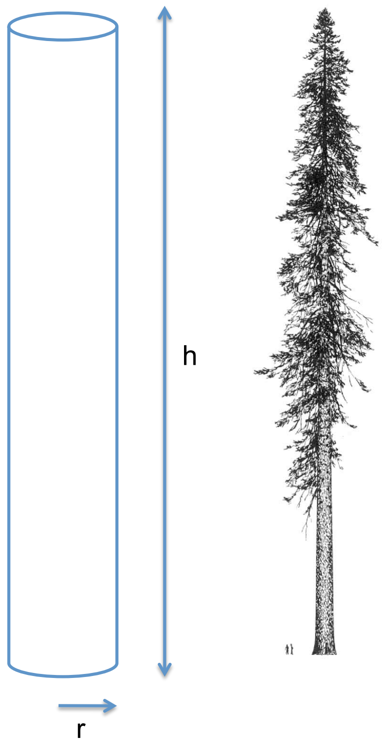
Trees achieve a large range of sizes, varying in radius from $0 - 4$ meters, in height from $0 - 98$ meters, and in mass from approximately 0.001 - 10000000 kg. But in spite of this wide range, trees seem to obey scaling laws: we do not expect to find a 50 m high tree with a 10 cm radius trunk, nor a 10 cm high tree with a 50m radius. As trees get wider, they also get taller, apparently in a very special way. Why is this? In this module we will explore how small trees become big through the framework of a scaling analysis.

There are many reasons for trees to become tall. One important reason is light competition: the tallest trees can capture the most light for photosynthesis, and prevent shorter trees from getting much light at all. This tall-strategy can increase reproductive success. However, being tall isn't always a good thing: tall trees must invest resources in building trunks, and may be at higher risk of dying in storms unless those trunks are sufficiently strong to survive damage. You should be able to think of many other costs and benefits of size that influence the scaling relationship between tree radius and tree height.

Independent of these evolutionary cost/benefit arguments, there are also physical laws that apply to trees that can constrain the scaling of radius and height. We are going to explore two mechanical principles today. So what could cause this tree to fail? To start off, let's simplify a tree as a column of radius $r$, height $h$, and density $\rho$. The total volume of our tree is then $\pi r^2 h$ and the total weight of the tree is $\rho \cdot g \cdot \pi r^2 h$, where $g$ is the acceleration of gravity ($9.8\ m \cdot s^{-2}$).

### 1.1 Compressive failure

First, let's imagine the tree were so heavy that it would be unable to support its own weight. You can imagine a tree made of Jello - being a rather weak material, a large enough column of the stuff would eventually cause the tree to collapse under itself. This is called **compressive failure**. Can this explain the scaling of tree radius and height?




Compressive failure occurs when the object's weight exceeds the maximum force the material can provide. Intuitively, an object with twice the area should be able to survive twice the force applied to it. This is true - this force is the product of the object's area and its maximum sustainable pressure, or **compressive strength**, which we will represent as $\sigma_c$. We can now write an equation for how big a tree has to be before compressive failure occurs. As before, the tree's weight is $\rho \cdot g \cdot \pi r^2 h$. The maximum compressive force is $\sigma_c \cdot \pi r^2$. Equating these two forces, we get

$$\begin{aligned} \rho \cdot g \cdot \pi r^2 h &= \sigma_c \cdot \pi r^2 \\ \rho \cdot g \cdot h &= \sigma_c \\ h &= \sigma_c / \rho g \end{aligned} \tag{1}$$

Something very interesting happened here - the tree radius is nowhere to be found in the final equation! If compressive failure alone controlled tree scaling, then tree height would have nothing to do with tree radius. The world would be filled with tall and skinny trees, tall and fat trees, short and skinny trees, and short and fat trees. Clearly there is more to the story than just compressive failure.

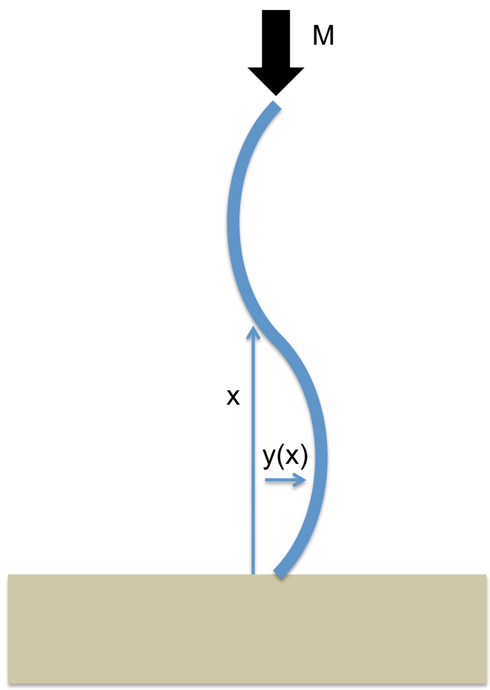
By the way, this equation does set an upper limit on how tall any tree could ever be.

For wood, $\sigma_c \approx 4.5 \times 10^7 \; kg \cdot m^{-1} \cdot s^{-2}$ and $\rho \approx 800 \; kg \cdot m^{-3}$. Substituting in these numbers, we obtain $h = 5739m$ - a tree almost 15 Empire State Buildings tall! This is a too large of a number - so what else could constrain tree size?

## 1.2 Buckling failure

Another way for a tree to fail is if its trunk bends and collapses. Imagine taping a plastic straw to a table so it is pointing upward. If you press down on the top end, the straw resists your force. But if you press down at any angle except the vertical, the straw bends and then becomes unstable, finally collapsing. This is **buckling**. This might also happen to trees - a gust of wind could apply a force that temporarily bends a tree, causing it to become unstable and collapse under itself. The difference between the straw and the tree is that in the first case, an external force caused the buckling to occur, while in the second case, the weight of the tree alone caused the buckling to occur.

In this section, we are going to find a relationship between the radius of a tree and the height at which it could spontaneously buckle.




Now consider a column that has bent a distance $y(x)$ perpendicular from the vertical x-axis due to some force applied to it. We can describe the bending of a column using the following approximate equation:

$$M = E \cdot I \cdot y''(x) \tag{2}$$

Here, $M$ is the applied moment. $E$ is the **Young's modulus (stiffness)** of the material and $I$ is the something called the second moment of area, which measures the bigness of the object. (And is not to be confused with the moment of inertia - the dimensionality is different. Oddly the same symbol is used for both.) Finally, $y''(x)$ is the second derivative of the displacement of the beam. You might notice that this equation, not accidentally, is similar in form to Newton's second law, $F = ma$. You can think of $M$ as the force $F$, $EI$ as the inertial mass $m$, and $y''(x)$ as the acceleration $a$.

By solving this equation we'll be able to understand how buckling might control tree radius and height. To start, the applied moment $M$ is defined as the product of a force and a distance. We're interested in the force of the tree's weight, applied at the distance the tree has bent. Thus,

$$M = -(\rho \cdot g \cdot \pi r^2 h) \cdot y(x) \tag{3}$$

(The negative sign comes because the force acts downward but the coordinate system increases upward). Also, we can look up what $I$ should be for a column of radius $r$. It turns out that

$$I = \frac{\pi r^4}{4} \tag{4}$$

Fortunately, $E$ is just a constant, so we'll leave it alone for now. Let's substitute these equations into our equivalent of Newton's second law:

$$\begin{aligned} M &= E \cdot I \cdot y''(x) \\ (-\rho \cdot g \cdot \pi r^2 h) \cdot y(x) &= E \cdot \tfrac{\pi r^4}{4} \cdot y''(x) \\ -\tfrac{4\rho g h}{E r^2} y(x) &= y''(x) \end{aligned} \tag{5}$$

Which can be rewritten in the standard form

$$\begin{gathered} y''(x) + k^2 y(x) = 0 \\ k = (\tfrac{4\rho g h}{E r^2})^{1/2} \end{gathered} \tag{6}$$

This ordinary differential equation has the solution

$$y(x) = A \sin kx + B \cos kx \tag{7}$$

for unknown constants $A$ and $B$. We're solving this equation for the the boundary conditions $y(x = 0) = 0$ and $y(x = h) = 0$ - that is, when the tree has bent, but the top and bottom ends have not yet moved. We know that once the tree bends, it collapses - so we only need to find a solution for this simple case! If we substitute the first boundary condition, we obtain $B = 0$. If we then substitute the second condition, we find that $A \sin kh = 0$. This equation holds when the sine function has zeros, or when

$$kh = n\pi \tag{8}$$

where $n$ can take on any non-negative integer value. $n = 0$ corresponds to an unbent tree; $n = 1$ corresponds to one bend, $n = 2$ to two bends, etc. Under the constraint that the tree's mass must be symmetrically distributed around the vertical, $n = 1$ is forbidden, so we will continue the analysis for $n = 2$.

Substituting in the definition of $k$ and using $n = 2$, the last equation becomes

$$\begin{aligned} (\tfrac{4\rho g h}{E r^2})^{1/2} h &= 2\pi \\ \tfrac{4\rho g h}{E r^2} h^2 &= 4\pi^2 \\ h^3 &= \tfrac{\pi^2 E}{\rho g} r^2 \\ h &= \left(\tfrac{\pi^2 E}{\rho g}\right)^{1/3} r^{2/3} \end{aligned} \tag{9}$$

This is the equation we want. Buckling sets a scaling relationship between height and radius. To avoid buckling, a tree must triple its radius in order to double its height. This is a situation of diminishing returns - tall trees require disproportionately large trunks to keep them from collapsing. Moreover, the overall scale of this relationship is set by the ratio of $E$ and $\rho g$. Trees can become taller if they become stiffer (increased $E$), lighter (lower $\rho$) or move to the Moon (lower $g$). This last option is sadly not possible! Note also that unlike with compressive failure, this equation does not set an absolute limit on tree height.

This equation looks complicated but can be rewritten to be more useful. If we log-transform both sides, and then substitute average values for wood ($E \approx 1.0 \times 10^{10}\ kg \cdot m^{-1} s^{-2}$, $\rho \approx 800\ kg \cdot m^{-3}$) we obtain

$$\begin{aligned} \log[h] &= \log\left[\left(\tfrac{\pi^2 E}{\rho g}\right)^{1/3} r^{2/3}\right] \\ \log[h] &= \log\left[\left(\tfrac{\pi^2 E}{\rho g}\right)^{1/3}\right] + \log[r^{2/3}] \\ \log[h] &= 5.44 + \tfrac{2}{3}\log[r] \end{aligned} \tag{10}$$

According to this equation, a plot of $\log[h]$ against $\log[r]$ should yield a straight line with slope 2/3 and intercept 5.44 when $r$ and $h$ are measured in meters.

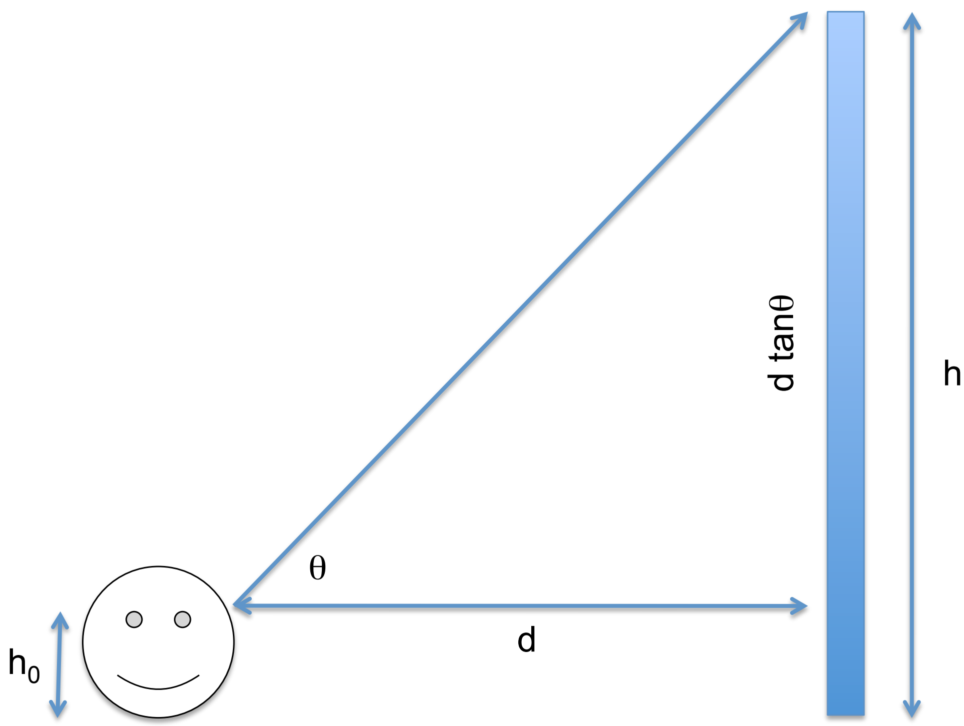
In reality, the mathematics that describe the buckling of a column due to its own weight are much more complex than what we've done here - we have linearized the curvature of the beam and neglected the uniform distribution of weight along the column. The References can point you to a more detailed solution. Interestingly, although the mathematics become far harder, the final prediction only changes in scale by 10-30%, with the 2/3 power scaling being preserved!

## 2 Experiment

We will test the buckling prediction by measuring the empirical scaling relationship between tree height and radius. You can measure tree radius by determining the circumference of a tree's trunk and dividing by $2\pi$. It is standard to measure the circumference at breast height, or at approximately 1 $m$ above the ground. You can measure tree height by standing a known distance $d$ away from a tree and measuring the angle $\theta$ from your eye to the top of the tree. Triangle trigonometry can show that the elevation from your eye to the tree top is $d \tan \theta$. If the height from the ground to your eye is $h_0$, then the total tree height is

$$h = h_0 + d \tan \theta \tag{11}$$

If you are standing on a slope, you will need to amend this equation to account for the incline. Fortunately there are not many hills on campus. Remember to take all measurements in meters!




We will provide you with a measuring tape and an angle measuring device (clinometer). You are responsible for creating a useful data sheet. Record the height and radius of as many different trees as you can. To avoid duplication of data, coordinate with other groups to choose unique locations to sample. We will create a group data file with everyone's measurements.

## 3 Analysis

Save the group data file as comma-separated-value file (CSV) with a header row. To load this data into MATLAB, run

```
data_trees = dlmread('trees.csv', ',', 1, 0);
radius_trees = data_trees(:,1);
height_trees = data_trees(:,2);
```

Now log-transform the data and plot it:

```
logradius_trees = log(radius_trees);
logheight_trees = log(height_trees);
hold all;
plot(logradius_trees, logheight_trees, 'xk');
```

We also need to calculate and plot the log-transformed buckling prediction:

```
E = 10 * 10^9;
rho = 800;
g = 9.8;
theory_intercept = log((pi*pi*E/(1*rho*g))^(1/3));
theory_slope = 2/3;
theory_xrange = log(linspace(min(radius_trees), max(radius_trees)));
theory_yrange = theory_intercept + theory_slope * theory_xrange;
plot(theory_xrange, theory_yrange, '-r');
```

We can be do better than squinting at this graph by also estimating the best-fitting line through the data. The most appropriate technique here is geometric (reduced major axis; RMA) regression. Make sure you have downloaded `gmregress.m` and placed it in your MATLAB working directory. Now use this code to estimate the slope and intercept of the best-fitting line and plot them overlaid on the data:

```
[coef coef_confidence_intervals] = gmregress(logradius_trees, logheight_trees);
regression_xrange = log(linspace(min(radius_trees), max(radius_trees)));
regression_yrange = coef(1) + coef(2) * regression_xrange;
plot(regression_xrange, regression_yrange, '-b');
```

You should directly examine the values stored in `coef` and `coef_confidence_intervals` to see how closely the data and the model match. Use the `xlabel`, `ylabel`, and `legend` (or other) commands to annotate the final graph.

Finally, download `bigtrees.csv`. This file contains data on height and radius from the National Register of Big Trees. Modify your code to also plot this data and the best-fitting line through it.

## 4 Discussion questions

1. How closely did the class's data match the `bigtrees` data set? How could you explain any differences?

2. Why do you think there is scatter in the data? How come all trees aren't on one single line?

3. Why is the buckling line higher than any of the data points? If it isn't, how is that possible?

4. Do you think buckling controls the scaling of tree size? What evidence do you have from your data to support your viewpoint?

5. What options do trees have to reduce their risk of buckling? Look back at the equation we derived - especially the part about the absolute scale factor.

6. We approximated a tree as a cylinder. What else might be important to understand how tree size scales?

7. When would it be useful to know how tree size scales? Are there scientific, commercial, or industrial applications of the relationship we derived? What other kinds of things could our buckling scaling relationship apply to? (Hint - think beyond biology!)

8. If elastic buckling doesn't set an upper limit to tree height, what does? Take a look at the Nature paper in the references.

## References


[1] National register of big trees. American Forests. Accessed 31 August 2010, `http://www.americanforests.org/resources/bigtrees/BigTreeRegister2010new.pdf`

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[7] Vogel, S. *Comparative Biomechanics.* Princeton University Press, 2003.

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